Matrix Fundamentals & Exercises

Due in class, Tuesday August 21st.

1. An $m \times n$ matrix is a 2-dimensional array, with m rows and n columns. Typically capital letters denote matrices, as in

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

2. A row vector is a 1×n matrix. Typically lowercase letters denote vectors

3. A column vector is a m×1 matrix. If not otherwise specified, a "vector" is usually a column vector.

4. The transpose of a matrix flips it about the diagonal and is denoted $^\top$. For example, if

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 8 & 5 \end{bmatrix}, \text{ then } A^\top = \begin{bmatrix} 3 & 4 & 8 \\ 1 & 2 & 5 \end{bmatrix}$$

5. Operations with scalars. A scalar (a single number) can be added or multiplied with a matrix in the natural way. That is, the operation is carried out on an element-by-element basis, as in

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \; 2 \times A = \begin{bmatrix} 2 \times 3 & 2 \times 4 \\ 2 \times 1 & 2 \times 2 \end{bmatrix}$$

6. Addition of matrices. Matrices can be added *only* if their dimensions match, and then the operation of addition is carried out on an element-by-element basis.

7. Inner or dot product of vectors is defined for two vectors of the sample length. The inner product is just the sum of the element-by-element product. For example

$$\begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} = 3 \times 4 \; + \; 8 \times 1 \; + \; 2 \times 5 = 30$$

8. Matrix multiplication is defined for a pair of matrices whose inner dimensions match. For example, if A is $n \times m$, and B is $m \times p$, then $A \times B$ is a valid matrix product. The result is a $n \times p$ matrix where each element is the dot product of a row of A and a column of B.

For example,

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 8 & 5 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 7 & 4 \end{bmatrix}$$

Note that A is $3{\times}2$ and B is $2{\times}3$, so matrix multiplication is valid between these two matrices. Then

$$A \times B = \begin{bmatrix} [3\ 1]\cdot[2\ 5] & [3\ 1]\cdot[1\ 7] & [3\ 1]\cdot[1\ 4] \\ [4\ 2]\cdot[2\ 5] & [4\ 2]\cdot[1\ 7] & [4\ 2]\cdot[1\ 4] \\ [8\ 5]\cdot[2\ 5] & [8\ 5]\cdot[1\ 7] & [8\ 5]\cdot[1\ 4] \end{bmatrix} = \begin{bmatrix} 11 & 10 & 7 \\ 18 & 18 & 12 \\ 41 & 43 & 28 \end{bmatrix}.$$

Note that for vectors, we now have an equivalent definition for a dot product. If x and y are two (column) vectors of the same length, then

$$x \cdot y = x^\top y.$$

9. Matrix inverse. The inverse of a square ($n{\times}n$) matrix A is written A^{-1}. Just as the multiplication of a number by it's inverse yields unity (e.g. $4 \times \frac{1}{4} = 1$), the matrix multiplication of A with A^{-1} yields something called an identity matrix. An identity matrix is a square matrix with all zeros except a diagonal of ones; an $n{\times}n$ identity matrix is written I_n.

   For a $3{\times}3$ matrix B, this looks like

$$B \times B^{-1} = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   There are no general explicit form for a matrix inverse, and often the inverse simply does not exist.

Exercises

Due in class, Tuesday August 21st.

Let $x = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Let $y = [1\ 5\ 3\ 6]$

Let $z = [4\ 2\ 1]$

Let $^\top$ denote the transpose operation, and let matrix multiplication be the default silent operator; that is, $xy = x \times y$.

1. Is the expression xy valid? If so, what is the value?
2. Is the expression $y^\top z$ valid? If so, what is the value?
3. Is the expression $x^\top z^\top$ valid? If so, what is the value?

Let $W = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

4. Is the expression xW valid? If so, what is the value?
5. Is the expression $Wz^\top$ valid? If so, what is the value?

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

6. Verify that $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
7. If $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, find B^{-1} and check your answer (i.e. compute BB^{-1})

TEN / August 20, 2001